Mathematical Methods II
Handout 21. Laurent Series.

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Laurent Series generalizes Taylor Series to include negative integers: $\sum_{n=-\infty}^{\infty} c_n(z-z_0)^n$. If we spell this out:

$$c_0 + c_1(z-z_0) + c_2(z-z_0)^2 + c_3(z-z_0)^3 + \cdots + \frac{c_{-1}}{z-z_0} + \frac{c_{-2}}{(z-z_0)^2} + \frac{c_{-3}}{(z-z_0)^3} + \cdots \tag{1}$$

A first important type of Laurent Series is to describe functions that are not analytic at a given point, about which one can not compute the Taylor Series. Let us remember the case of the function $f(z) = \exp(-1/z^2)$. It is analytic everywhere but at the origin. For any nonzero z, we can formally represent the Series representation of the exponential $\exp(z) = \sum_{k=0}^{\infty} z^k/k!$ and substitute z for $-1/z^2$. This gives us the series expansion:

$$f(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}\frac{1}{z^{2k}} = 1 - \frac{1}{z^2} + \frac{1}{2z^4} - \frac{1}{6z^6} + \cdots \tag{2}$$

which is a Laurent Series, convergent for all $|z| > 0$. Not all Series are like this, of course, for instance:

$$\frac{e^z}{z^3} = \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{2z} + \frac{1}{3!} + \frac{z}{4!} + \frac{z^2}{5!} + \cdots \tag{3}$$

has a finite number of terms with negative powers. We will see in next lecture how such Laurent expansions allow to classify singularities.

There can be various domains of convergence for a Laurent Series. They are annuli of the type:

$$\mathcal{A}(z_0, r, R) = \{z \,:\, r < |z| < R\} \tag{4}$$

and the expression of the Laurent Series on various annuli can be different, although the Laurent expansion of an analytic function on a given annulus is unique (this is important for actual calculation since one can obtain an expression combining various tricks and whenever a result is obtained, we know this is the one). This can be illustrated with an example. Consider the function $f(z) = 3/(2+z-z^2)$. The denominator factors out as $(1+z)(2-z)$ so the partial fraction decomposition yields:

$$f(z) = \frac{1}{z+1} + \frac{1}{2}\frac{1}{1-z/2} \tag{5}$$

This gives us four possible series decompositions: in and out the circle of convergences:

$$\frac{1}{1+z} = \sum_{k=0}^{\infty}(-z)^k\,,\ \text{for } |z| < 1\,, \tag{6a}$$

$$\frac{1}{1-z/2} = \sum_{k=0}^{\infty}(z/2)^k\,,\ \text{for } |z| < 2\,, \tag{6b}$$

$$\frac{1}{1+z} = \frac{1}{z}\frac{1}{1+\frac{1}{z}} = \sum_{k=0}^{\infty}\frac{(-1)^k}{z^{k+1}}\,,\ \text{for } |z| > 1\,, \tag{6c}$$

$$\frac{1}{1-z/2} = \frac{2}{z}\frac{-1}{1-2/z} = -\sum_{k=0}^{\infty}\frac{2^{k+1}}{z^{k+1}}\,,\ \text{for } |z| > 2\,, \tag{6d}$$

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yielding three types of Laurent Series. In the unit disk, we sum Eqs. (6a) and (6b):

$$f(z) = \sum_{k=0}^{\infty}((-1)^k + (1/2)^{k+1})z^k = \frac{3}{2} - \frac{3}{4}z + \frac{9}{8}z^2 - \frac{15}{16}z^3 + \cdots \quad \text{for } |z| < 1\,, \tag{7}$$

which is the Maclaurin Series. In the annulus $\mathcal{A}(0,1,2)$, we sum Eqs. (6b) and (6c):

$$f(z) = \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{z^k} + \sum_{k=0}^{\infty}\frac{z^k}{2^{k+1}} = \cdots + \frac{1}{z^3} - \frac{1}{z^2} + \frac{1}{z} + \frac{1}{2} + \frac{z}{4} + \frac{z^2}{8} + \cdots \quad \text{for } 1 < |z| < 2\,, \tag{8}$$

and for $|z| > 2$, i.e., in the annulus $\mathcal{A}(0,2,\infty)$, summing Eqs. (6c) and (6d):

$$f(z) = \sum_{k=1}^{\infty}\frac{(-1)^{k-1} - 2^{k-1}}{z^k} = -\frac{3}{z^2} - \frac{3}{z^3} - \frac{9}{z^4} - \frac{15}{z^5} - \cdots\,, \quad \text{for } |z| > 2\,. \tag{9}$$

When the inner circle can be shrinked to a single point to exclude only one point, which was the case of our two first examples, the series of the negative terms is called *the principal part* of f at z_0.

There is a Laurent theorem analogous to the Taylor theorem that provides the Taylor expansion of an analytic function. It states that an analytic function f in a domain that contains two concentric circles $\mathcal{C}_1$ and $\mathcal{C}_2$ with center z_0 can be represented in the annulus that separates them as: $f(z) = \sum_{n=-\infty}^{\infty}\left[\frac{1}{2i\pi}\oint_{\mathcal{C}}\frac{f(w)}{(w-z_0)^{n+1}}\,dw\right](z-z_0)^n$, with $\mathcal{C}$ any single closed path within the annulus. The proof is an extension of the proof of Taylor's theorem and is left as a problem.

We conclude with a powerful result on the division of power series: if f and g are two analytic functions at z_0 and $g(z_0) \neq 0$, with Taylor series representations $f(z) = \sum_{k=0}^{\infty} a_k(z-z_0)^k$ and $g(z) = \sum_{k=0} b_k(z-z_0)^k$, then the quotient f/g has the Taylor series $\frac{f(z)}{g(z)} = \sum_{k=0}^{\infty} c_k(z-z_0)^k$ where the coefficients c_k satisfy the long division algorithm:

$$a_n = b_0c_n + \cdots + b_nc_0\,. \tag{10}$$

A. Suggested readings

- `http://www.mathsisfun.com/algebra/partial-fractions.html` or `http://goo.gl/jKmK7u` on partial fraction decomposition.
- `http://en.wikipedia.org/wiki/Long_division`.

B. Exercises

1. Find all the Laurent Series of $1/(z^3 - z^4)$ with center 0.
2. Find all the Laurent Series of $(1+z)^2/(1+z+z^2)$ with center 0.
3. Find the Laurent Series of $\cos(z)/z^4$, e^{z^2}/z^3, $z^3\cosh(1/z)$.
4. Find the Laurent Series of $1/(1-z)$ and $1/z^2$ centered at $z_0 = i$.
5. Find the series expansion of $1/(z^2+4)$ valid in the region $|z - 2i| > 4$.

C. Problems

1. Show that $\cosh(z + 1/z) = \sum_{k=-\infty}^{\infty} c_k z^k$ with coefficients:

$$c_k = \frac{1}{2\pi}\int_0^{2\pi}\cos(n\theta)\cosh(2\cos(\theta))\,d\theta\,. \tag{11}$$

   Hint: use contour integration on the unit disk.
2. Prove Laurent's theorem (use Taylor theorem as a guide).