MÉTODOS MATEMÁTICOS II
Lecture 13: The Cauchy-Goursat theorem and its integral forms.

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(Dated: February 25, 2013)

We have proved Cauchy's theorem through Green's theorem. A more general proof, due to Goursat, relaxes the condition of continuity of the derivative. It is proven on a triangle ABC cut through its mid-points (E between AB, F between BC and D between AC):

$$\oint_\Delta = \oint_{ABCA} = \int_{DAE} + \int_{EBF} + \int_{FCD} \tag{1}$$

$$= \left[\int_{DAE} + \int_{ED}\right] + \left[\int_{EBF} + \int_{FE}\right] + \left[\int_{FCD} + \int_{DF}\right] + \left\{\int_{DE} + \int_{EF} + \int_{FD}\right\} \tag{2}$$

$$= \oint_{DAED} + \oint_{EBFE} + \oint_{FCDF} + \oint_{DEFD}\,. \tag{3}$$

which yields, through iterations:

$$\left|\oint_\Delta f(z)\,dz\right| \le 4^n \left|\oint_{\Delta_n} f(z)\,dz\right|\,, \tag{4}$$

where Δ_n is the nth iterated triangle obtained by taking successive mid-points of Δ_{n-1} ($\Delta_0 = \Delta$). With P the perimeter of Δ, assuming differentiability of f, we arrive at $\left|\oint_\Delta f(z)\,dz\right| \le \epsilon P^2$ for any $\epsilon > 0$, proving that the contour integral vanishes.

The main consequences of Cauchy's theorem is **Cauchy's integral theorem**: an holomorphic function f in a simply connected domain D satisfy, for all $z_0 \in D$ and any simple closed path $\mathcal{C}$ that encloses z_0:

$$f(z_0) = \frac{1}{2i\pi}\oint_{\mathcal{C}} \frac{f(z)}{z - z_0}\,dz\,. \tag{5}$$

This is useful to compute contour integrals. For instance:

$$\oint \frac{ie^z}{z-1}\,dz = -2\pi\exp(1) \approx -17\,, \tag{6}$$

The Cauchy's integral formula also proves that holomorphic are infinitely differentiable:

$$f'(z_0) = \frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{f(z)}{(z-z_0)^2}\,dz \tag{7}$$

$$f''(z_0) = \frac{2!}{2\pi i}\oint_{\mathcal{C}} \frac{f(z)}{(z-z_0)^3}\,dz \tag{8}$$

$$\dots \tag{9}$$

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_{\mathcal{C}} \frac{f(z)}{(z-z_0)^{n+1}}\,dz \tag{10}$$

$$\tag{11}$$

which is also useful to compute contour integrals. For instance,

$$\oint_{\mathcal{C}} \frac{\cos z}{(z-i\pi)^2}\,dz = -2i\pi\sin(i\pi) = 2\pi\sinh(\pi) \approx 72.6\,, \tag{12}$$

for $\mathcal{C}$ encircling $i\pi$, zero otherwise.

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A. Suggested readings

- `http://en.wikipedia.org/wiki/Cauchy's_integral_formula`
- "A Simple Proof of the Fundamental Cauchy-Goursat Theorem", E. H. Moore, Trans. Amer. Math. Soc. **1**, 499 (1900) `doi:10.2307/1986368` or `http://goo.gl/ZJXoa`.
- `http://laussy.org/wiki/MMI/The_Cauchy-Goursat_theorem_and_its_integral_forms` at `http://goo.gl/hhIX7` for detailed proofs of the theorems.

B. Exercises

1. Show that f being analytic for $\oint f(z)\,dz = 0$ is a sufficient but not necessary condition.
2. Compute the following integrals on a path that encloses the singularity in each case:

$$\oint_{\mathcal{C}} \frac{\sin(z)}{z}\,dz\,,\quad \oint_{\mathcal{C}} \frac{z^3-6}{2z-i}\,dz\,,\quad \oint_{\mathcal{C}} \frac{e^{2z}}{\pi z-i}\,dz\,,\quad \oint_{\mathcal{C}} \frac{z^2\sin(z)}{4z-1}\,dz \quad\text{and}\quad \oint_{\mathcal{C}} \frac{e^z}{ze^z-2iz}\,dz\,. \tag{13}$$

3. Compute $\int_{\mathcal{C}} \frac{z^2+1}{z^2-1}\,dz$ on $\mathcal{C} = \{z\,:\,|z-1|=0\}$, $\{z\,:\,|z+1|=0\}$, $\{z\,:\,|z|=1/2\}$ and $\{z\,:\,|z|=2\}$.
4. Compute for $\mathcal{C} = \{z\,:\,|z|=3/2\}$:

$$\oint_{\mathcal{C}} \frac{z^4-3z^2+6}{(z+i)^3}\,dz \quad\text{and}\quad \oint_{\mathcal{C}} \frac{\exp(z)}{(z-1)^2(z^2+4)}\,dz\,. \tag{14}$$

C. Problems

1. Check that the Cauchy-Goursat demonstration fails for z^*.
2. Prove Eq. (5) by deforming $\mathcal{C}$ to a circle centered on z_0 and the change of variable $z-z_0=\epsilon e^{i\theta}$ for a circle of vanishing radius.
3. Prove Cauchy-Goursat's theorem for a polygon and a simple closed curve.